Effects of the Generalized Uncertainty Principle on the Inflation Parameters

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Abstract

We investigate the effects of the generalized uncertainty principle on the inflationary dynamics of the early universe in both standard and braneworld viewpoint. We choose the Randall-Sundrum II model as our underlying braneworld scenario. We find that the quantum gravitational effects lead to a spectral index which is not scale invariant. Also, the amplitude of density fluctuations is reduced by increasing the strength of quantum gravitational corrections. However, the tensor-to-scalar ratio increases by incorporation of these quantum gravity effects. We outline possible manifestations of these quantum gravity effects in the recent and future observations.

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1 Introduction

One of the most important achievements in the spirit of quantum gravity theories such as string theory, is that the laws of physics are considerably different at short distances [1-9]. For example, the standard quantum mechanical commutation relations change to some modified (generalized) commutators at the length scale of the order of Planck length [10-11]. These modifications are negligible in low energy physics but at high energy such as the early universe, quantum gravitational corrections play a crucial role [12]. As a result, the standard uncertainty relation of quantum mechanics is replaced by a gravitational uncertainty relation which contains a minimal observable length of the order of the Planck length [13]. The very notion of spacetime in this quantum gravity era cannot be probed more precisely than this minimal observable length [1,2,11]. These fundamental properties of spacetime in quantum gravity era result in a variety of phenomenologically interesting outcomes for the rest of physics (see for instant [14]). In this paper we investigate the effects of the generalized uncertainty principle on the inflationary parameters in the standard and braneworld inflation. The braneworld scenario we analyzed is the Randall-Sundrum II setup which is one of the most promising alternatives of extra dimensional theories [15]. One way to discover the quantum gravitational effects in the inflationary era is to study the perturbation spectrum generated during inflation. Any modifications to the scalar and tensorial perturbations spectrum due to these quantum gravity effects are essentially detectable in the cosmic microwave background (CMB) data [16-22]. This provides a reliable approach to test the theories of short distance physics. In which follows a prime on a quantity denotes differentiation with respect to its argument.

2 Inflation and the Generalized uncertainty principle

In short distances the standard commutation relations will be changed as [11,13,14]

$$[x_i, p_j] = i\hbar(\delta_{ij} + \beta p^2 \delta_{ij} + \beta' p_i p_j), \tag{1}$$

$$[p_i, p_j] = 0, \tag{2}$$

and

$$[x_i, x_j] = i\hbar \frac{(2\beta - \beta') + (2\beta + \beta')\beta p^2}{(1 + \beta p^2)}(p_i x_j - p_j x_i). \tag{3}$$

We set $\beta' = 0$, so the corresponding Poisson brackets are

$$\{x_i, p_j\} = \delta_{ij}(1 + \beta p^2), \quad \{p_i, p_j\} = 0, \quad \{x_i, x_j\} = 2\beta(p_i x_j - p_j x_i) \tag{4}$$

The parameter β is related to the minimum length *i.e.* $x_{min} \sim \sqrt{\beta}$. The calculation of inflationary scalar density perturbations in the presence of the minimal length are preformed in Ref. [23]. The scalar field, ϕ which drives the inflation has energy density and pressure

$$\rho = \frac{1}{2}\dot{\phi}^2 + V$$
$$p = \frac{1}{2}\dot{\phi}^2 - V \tag{5}$$

respectively where $V(\phi)$ is the inflation potential. The slow-roll parameters are given as usual by [24]

$$\begin{aligned}
\epsilon &= \frac{M_4^2}{2}\Big(\frac{V'}{V}\Big)^2 \qquad (6)\\
\eta &= \frac{M_4^2}{2}\frac{V''}{V}\ .
\end{aligned}$$

In the slow-roll regime we have

$$\begin{aligned}
\frac{1}{2}\dot{\phi}^2 &\ll V(\phi)\\
3H\dot{\phi} &\simeq -V'(\phi) \qquad (7)
\end{aligned}$$

We now proceed to incorporate the quantum gravitational corrections as are given by equations (1)-(3) within a typical inflationary scenario. One could assume that there is a fundamental energy scale Λ (Planck or string scale) that these corrections become important. Defining the conformal time as [25]

$$\tau = -\frac{1}{aH}\ . \qquad (8)$$

We will see that physical momentum p and the comoving momentum k are related through

$$k = ap = -\frac{p}{\tau H}\ . \qquad (9)$$

The conformal time that the new physics rules prior to it is given by

$$\tau_0 = -\frac{\Lambda}{Hk} \qquad (10)$$

where Λ is the Planck energy scale. Using equation (1), we change the comoving momentum k before the time τ_0 to $k(1+\beta k^2)$. This is a realization of the modified dispersion relation supported by loop quantum gravity and noncommutative geometry [26,27]. The equation governing the evolution of perturbations in the inflation era is [24]

$$\mu_k'' + \Big(k^2 - \frac{a''}{a}\Big)\mu_k = 0 \qquad (11)$$

where μ is a rescaled field $\mu = a\delta\phi$ and a prime denotes differentiation with respect to τ. The scalar spectral index in the presence of the minimal length cutoff now is given by

$$n_s - 1 = \frac{d\ln\mathcal{P}_s}{d\ln k(1+\beta k^2)} \qquad (12)$$

where $\mathcal{P}_s$ is the amplitude of the scalar perturbation. Therefore, we find

$$n_s = \frac{1+\beta k^2}{1+3\beta k^2}\frac{d\ln\mathcal{P}_s}{d\ln k} + 1 \simeq (1-2\beta k^2)\frac{d\ln\mathcal{P}_s}{d\ln k} + 1\ . \qquad (13)$$

As an important result, the spectral index is not scale invariant in this case. Any deviation from scale invariance of the spectral index essentially contains a footprint of these quantum gravity effects. The change in the Hubble parameter due to the modified commutators will be realized using slow-roll parameters. At the horizon crossing epoch we have [24,25]

$$\frac{dH}{dk} = -\frac{\epsilon H}{k} \tag{14}$$

changing k to $k(1+\beta k^2)$ we find

$$H \simeq k^{-\epsilon} e^{-\beta\epsilon k^2} \; . \tag{15}$$

Using equation (11), the tensorial density fluctuation is given by

$$\mathcal{P}_t(k) = \frac{1}{a^2} <\mid \mu_k(\tau) \mid^2> \tag{16}$$

Following [25], the tensorial density fluctuations in our case is

$$\mathcal{P}_t(k) = \Big(\frac{H}{2\pi}\Big)^2 \Big(1 - \frac{H}{\Lambda}\sin(\frac{2\Lambda}{H})\Big), \tag{17}$$

where the second term on the right hand side is a direct contribution of quantum gravity effect. For scalar density fluctuations, one should add an extra $(\frac{H}{\dot{\phi}})^2$ term [24,25]

$$\mathcal{P}_s(k) = \Big(\frac{H}{\dot{\phi}}\Big)^2 \Big(\frac{H}{2\pi}\Big)^2 \Big(1 - \frac{H}{\Lambda}\sin(\frac{2\Lambda}{H})\Big) \tag{18}$$

where H is given by equation (15). Figure 1 shows the k-dependance of tensorial fluctuations for a fixed ϵ and β while figure 2 shows the β-dependance of it for a fixed k. As usual, Λ is chosen to be the Planck energy scale. We note that by variation of β (which is essentially a fixed quantity related to the minimal observable length) we just mean a control on the strength of the quantum gravity effect. The tensor-to-scalar ratio in this setup is given by

$$\frac{\mathcal{P}_t}{\mathcal{P}_s} = \Big(\frac{\dot{\phi}}{H}\Big)^2 = \Big(\frac{16\pi\sqrt{\epsilon}V}{M_4 k^{-\epsilon} e^{-\beta\epsilon k^2}}\Big)^2, \tag{19}$$

where M_4 is the 4-dimensional fundamental scale. The extra factor $e^{-\beta\epsilon k^2}$ in the denominator is the correction due to the GUP effects. Figure (2) shows the difference of tensor-to-scalar ratio in standard and modified cases. As we see, the tensor-to-scalar ratio increases by incorporation of the quantum gravity effects.

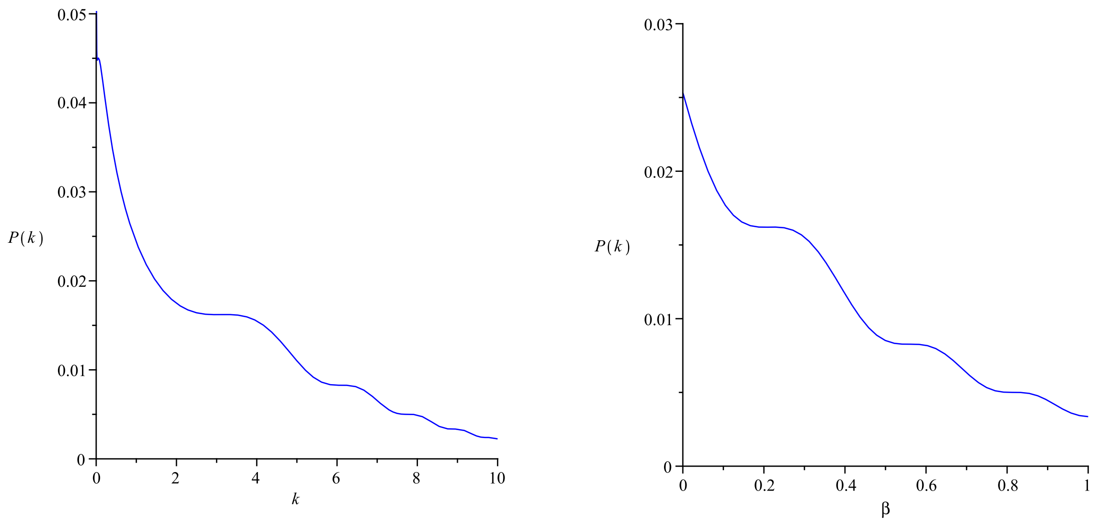

Figure 1: a) k-dependance of tensorial density fluctuations for $\epsilon = \beta = 0.1$ b) β-dependance of tensorial density fluctuations for $k = 1$. Here there is an oscillatory behavior which can be detected essentially in the CMB spectrum as a trace of quantum gravity effects.

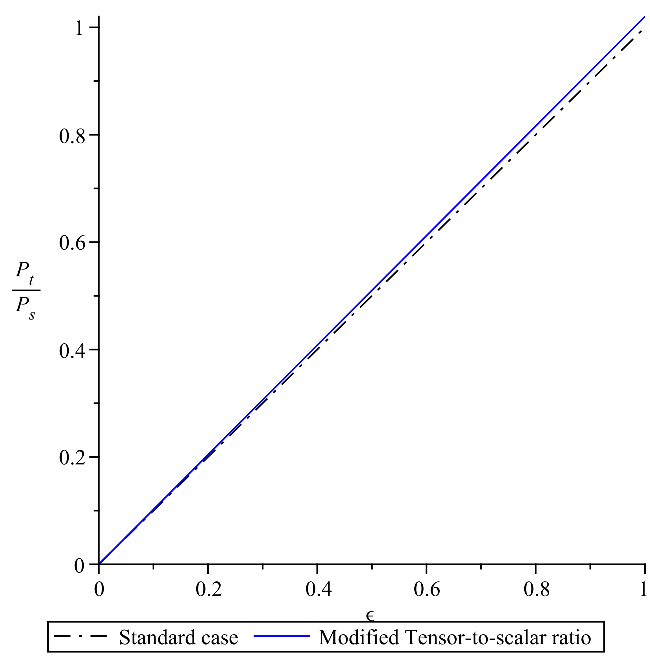

Figure 2: Difference of tensor-to-scalar ratio for standard and GUP modified inflation for a fixed k and $\beta = 0.01$

3 Brane inflation

Now we turn our attention to a braneworld inflation scenario. We study the case of Randall-Sundrum II setup [15] in the presence of the generalized uncertainty principle. The Randall-Sundrum II model has a single positive tension brane living in an infinite AdS bulk. Modified Friedmann equation for this setup is given by [28]

$$H^2 = \frac{\Lambda_4}{3} + \Big(\frac{8\pi}{3M_4^2}\Big)\rho + \Big(\frac{4\pi}{3M_5^3}\Big)^2\rho^2 + \frac{\mathcal{E}}{a^4} \tag{20}$$

where $\mathcal{E}$ is an integration constant originating from the projection of the bulk Weyl tensor. The last term is called the dark radiation and we neglect it in our analysis because during inflation it will vanish really fast. Also neglecting the cosmological constant term in the early universe we can rewrite the Friedmann equation as

$$H^2 = \frac{8\pi}{3M_4^2}\rho\Big[1 + \frac{\rho}{2\lambda}\Big] \tag{21}$$

where λ is the brane tension. In the inflation era, the only matter on the brane is a scalar field, ϕ with energy density and pressure defined as (5).

In this braneworld inflation scenario, the slow-roll parameters can be defined as [29]

$$\begin{aligned} \epsilon &\equiv \frac{M_4^2}{16\pi}\Big(\frac{V'}{V}\Big)^2\Big[\frac{2\lambda(2\lambda+2V)}{(2\lambda+V)^2}\Big] \\ \eta &\equiv \frac{M_4^2}{8\pi}\Big(\frac{V''}{V}\Big)\Big[\frac{2\lambda}{2\lambda+V}\Big]. \end{aligned} \tag{22}$$

The amplitude of scalar perturbations in the slow-roll limit in this case is [24,30]

$$\mathcal{P}_s = \frac{9}{25}\frac{H^6}{V'^2}. \tag{23}$$

Now we consider the effects of the generalized uncertainty principle. Using equations (15) and (23) we find

$$\mathcal{P}_s = \frac{9}{25V'^2}\Bigg[k^{-\frac{M_4^2}{16\pi}\left(\frac{V'}{V}\right)^2\left[\frac{2\lambda(2\lambda+2V)}{(2\lambda+V)^2}\right]}\exp\Big\{-\beta\Big(\frac{M_4^2}{16\pi}\Big[\frac{V'}{V}\Big]^2\Big[\frac{2\lambda(2\lambda+2V)}{(2\lambda+V)^2}\Big]\Big)k\Big\}\Bigg]^6. \tag{24}$$

We choose the inflaton field potential to be chaotic type *i.e.* $V(\phi) = \frac{1}{2}m^2\phi^2$ so equation (24) becomes

$$\mathcal{P}_s = \frac{9}{25m^4\phi^2}\Bigg[k^{-\frac{M_4^2}{4\pi\phi^2}\left[\frac{2\lambda(2\lambda+m^2\phi^2)}{(2\lambda+\frac{1}{2}m^2\phi^2)^2}\right]} \times e^{-\beta\left(\frac{M_4^2}{4\pi\phi^2}\left[\frac{2\lambda(2\lambda+m^2\phi^2)}{(2\lambda+\frac{1}{2}m^2\phi^2)^2}\right]\right)k}\Bigg]^6 \tag{25}$$

Figure (2a) shows the scalar density perturbations against k for given values of other parameters while figure (2b) shows its behavior in terms of β. We see that relative to standard case, $\mathcal{P}_s$ reduces by incorporation of quantum gravity effects. Also, $\mathcal{P}_s$ reduces by enhancement of the role played by quantum gravity effects.

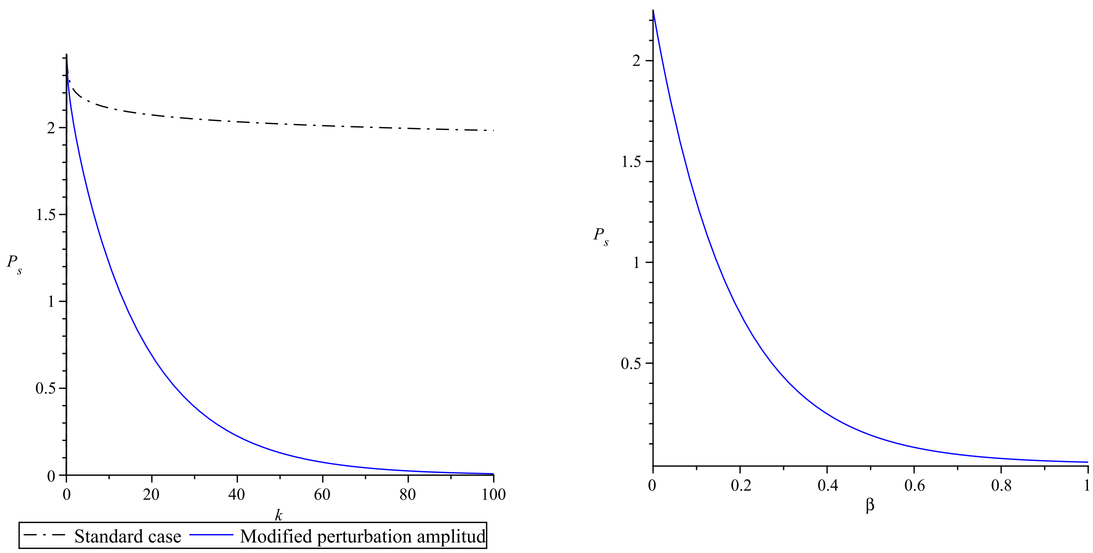

Figure 3: a) k-dependance of scalar density fluctuations for $\beta = 0.01$, $M_4 = 1$, $m = 0.1M_4$, $\lambda = 0.1M_4$ and $\phi = 4M_4$. b) β-dependance of scalar density fluctuations for $k = 1$, $M_4 = 1$, $m = 0.1M_4$, $\lambda = 0.1M_4$ and $\phi = 4M_4$.

Now the scalar spectral index is

$$n_s - 1 \equiv \frac{d \ln \mathcal{P}_s}{dk} \tag{26}$$

Using equation (25) we plot the spectral index against k and β. Figure 3 shows the result of this calculation. If the quantum gravity effects are relatively small, the spectral index shows a scale invariance behavior. However, in the limit of strong quantum gravity effects, the spectral index has no scale invariance characteristics.

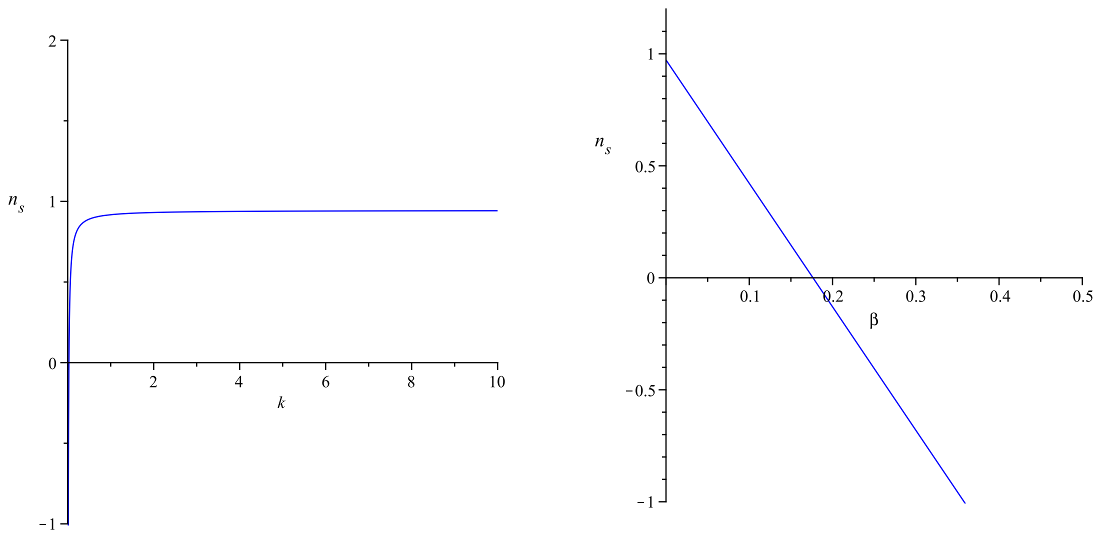

Figure 4: a) k-dependance of the scalar spectral index for $\beta = 0.01$, $M_4 = 1$, $m = 0.1M_4$, $\lambda = 0.1M_4$ and $\phi = 4M_4$. b) β-dependance of the scalar spectral index for $k = 1$, $M_4 = 1$, $m = 0.1M_4$, $\lambda = 0.1M_4$ and $\phi = 4M_4$.

4 Conclusion

In this letter we have considered the effects of the generalized uncertainty principle (as a common feature of all promising quantum gravity candidates) on the inflationary dynamics of both the standard $4D$ theory and the Randall-Sundrum II braneworld setup. As an important result, we have shown that in the presence of the strong quantum gravity effects, the spectral index is not scale invariant. In this sense, any deviation from scale invariance of the spectral index essentially contains a footprint of these high energy effects. There is an oscillatory behavior in the k-dependance of density fluctuations which can be detected essentially in the CMB spectrum as a trace of these effects. The tensor-to-scalar ratio increases by incorporation of the quantum gravity effects. In a braneworld viewpoint of inflation on the Randall-Sundrum II brane, we have shown that relative to the standard case, $\mathcal{P}_s$ reduces by incorporation of the generalized uncertainty principle. Also, $\mathcal{P}_s$ reduces by enhancement of the role played by quantum gravity effects. In this case, similar to $4D$ case, only in the limit of week quantum gravity effects the spectral index shows a scale invariant behavior. However, in the limit of strong quantum gravity effects, the spectral index has no scale invariance properties. We note that any modifications to the scalar and tensorial perturbations spectrum (such as oscillatory behavior in the amplitude of fluctuations) due to these quantum gravity effects are essentially detectable in the spectrum of the cosmic microwave background radiation and this may provide an indirect test of quantum gravity proposals.

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